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# Rearrangements of Fourier Series*

SZILÁRD GY. RÉVÉSZ

*Department of Analysis, Eötvös University, Muzeum krt. 6–8, Budapest H-1088, Hungary*

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## 1

In the present work our attention is paid primarily to continuous functions and their classical trigonometrical Fourier series, though we shall also prove some more general theorems. Let us denote by $T$ the one-dimensional torus $\mathbb{R}/2\pi\mathbb{Z}$, and, for $f \in L(T)$, write for its Fourier series

$$\begin{aligned} f(x) &\sim \sum_{k=0}^{\infty} a_k \cos kx + b_k \sin kx \\ &= \sum_{k=0}^{\infty} A_k(x) \qquad (b_0 := 0). \end{aligned} \tag{1.1}$$

The conjugate function $\tilde{f}$ is the function with Fourier series

$$\begin{aligned} \tilde{f}(x) &\sim \sum_{k=0}^{\infty} b_k \cos kx + a_k \sin kx \\ &= \sum_{k=0}^{\infty} B_k(x) \qquad (B_0(x) \equiv 0). \end{aligned} \tag{1.2}$$

Put

$$p_k^{(n,m)} = \begin{cases} 1, & k \leqslant n, \\ 1 - \dfrac{k-n}{m}, & n \leqslant k \leqslant n+m. \end{cases} \tag{1.3}$$

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The partial sums, the Fejér means, and the de la Vallée Poussin means of $f$ are

$$\begin{aligned} S_n(f)(x) &= \sum_{k=0}^{n} A_k(x) \\ \sigma_n(f)(x) &= \frac{1}{n}\sum_{k=0}^{n-1} S_k(f)(x) = \sum_{k=0}^{n}\left(1-\frac{k}{n}\right)A_k(x), \\ V_{n,m}(f)(x) &= \frac{1}{m}\sum_{k=n}^{n+m-1} S_k(f)(x) \\ &= \frac{n+m}{m}\sigma_{n+m}(f)(x) - \frac{n}{m}\sigma_n(f)(x) = \sum_{k=0}^{n+m} p_k^{(n,m)} A_k(x). \end{aligned} \tag{1.4}$$

For various subspaces of $C(T)$ many results are known concerning the uniform convergence of Fourier series. But in general, Fourier series will not converge (in the natural sense of convergence in $C(T)$, that is, uniformly). The subspace of functions having uniformly convergent expansion (1.1) is not a nice subspace in $C(T)$. For example, there are examples of Salem [13] and Kahane and Katznelson [5] that $S_n(|f|)$ or $S_n(f^2)$ may diverge though $S_n(f) \to f$ uniformly. The only positive result with unrestricted generality is a strange-looking result of Menšov [7], asserting that every $f \in C(T)$ can be decomposed as $f = f_1 + f_2$ so that there exist some $n_k^{(i)} \to \infty$ $(i = 1, 2)$ for which $S_{n_k^{(i)}}(f) \to f$ uniformly in $T$ $(i = 1, 2)$. However, the subsequences $n_k^{(i)}$ of $\mathbb{N}$ are different in general.

Of course, a.e. convergence follows from the theorem of Carleson [2], but Menšov [7] showed that there is a $f \in C(T)$ for which for any given $n_k \to \infty$, $S_{n_k}(f)(x_0)$ diverges for some $x_0$ (depending on $f$ and $n_k$). This excludes the everywhere pointwise, and so also the uniform convergence of any subsequence of the partial sums. By a certain delicate construction on the basis of Fejér's example, Busko [1] showed that for each $\omega(n) = o(\log n)$ there exists some $f \in C(T)$ so that

$$\frac{\|S_n(f)\|_\infty}{\omega(n)} \to \infty. \tag{1.5}$$

Since for $f \in C(T)$, $\|S_n(f)\|_\infty = o(\|S_n\|)$, where $S_n$ is the $n$th partial sum operator, and its usual operator norm $\|S_n\| = (4/\pi)\log n + O(1)$ (see [10, p. 67]), this result of Busko is sharp.

With these negative results in mind, we investigate rearrangements. For a permutation or rearrangement $\nu: \mathbb{N} \leftrightarrow \mathbb{N}$, write

$$f(x) \overset{\nu}{\sim} \sum_{k=0}^{\infty} A_{\nu(k)}(x). \tag{1.6}$$

The corresponding partial sums will be denoted by ${}_{\nu}S_n(f)(x)$, or in short ${}_{\nu}S_n$. Note that ${}_{\nu}S_n$ as an operator has norm

$$\|{}_{\nu}S_n\| = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{k=0}^{n}\cos \nu(k)t\right| dt > C\cdot\log n, \tag{1.7}$$

by the solution of a famous problem of Littlewood (see the independent proofs [4, 6]). This means that by rearrangement we can gain nothing regarding the operator norm.

Observe that (1.7) implies via elementary calculations or the Banach–Steinhaus theorem that for any universally prescribed rearrangement $\nu$ and subsequence $n_k \to \infty$ there exists an $f\in C(T)$ such that ${}_{\nu}S_{n_k}(f)$ diverges to an extent similar to (1.5). That is, we have

$$\sup_k \frac{\|{}_{\nu}S_{n_k}(f)\|_\infty}{\omega(n_k)} = \infty.$$

This means that something must depend on $f$.

The second question would be to find a universal $\nu$ and for all $f\in C(T)$ an $n_k(f)$ with ${}_{\nu}S_{n_k}(f) \to f$ uniformly. Though the constructions of Busko and Menšov cannot be trivially transplanted to the case of (1.6), we doubt the possibility of finding such a universal rearrangement.

So we can only hope to prove that for all $f\in C(T)$ there exists a $\nu$ with ${}_{\nu}S_n(f)\to f$ uniformly, which is formulated in Section 4 as a conjecture.

On the way to deciding this problem, we have the following result.

THEOREM 1. *For any $f\in C(T)$ there exist some $\nu$ and $n_k$ (both depending on $f$) such that ${}_{\nu}S_{n_k}(f)\to f$ uniformly in $T$.*

As for the proof of this theorem, the key part of that and of the whole paper, as well, is Lemma 2. It uses a rather elementary probabilistic construction to find a block of the wanted $\nu$. The underlying idea could be interpreted as a simulation of the nice, convergent, and positive method of $(C, 1)$ summation by constructing appropriate blocks of $\nu$.

In proving Theorem 1, we prove Theorem 2 in Section 2, which enables us to deduce a somewhat curious theorem concerning the speed of the approximation by ${}_{\nu}S_{n_k}(f)$. Theorem 3 in Section 5 means that for "weakly continuous" functions, i.e., which have a relatively large modulus of continuity, our method of rearrangement gives in some sense the best approximation.

Section 6 is devoted to technical strengthenings of Theorem 1. We can prove several relations concerning the "smoothness" of the permutation, the "slow increase" of $n_k$, and an estimate from below of the number of $n_k$

with $n_k \leqslant x$. Further, we show that the only good thing we had in the original case, i.e., the a.e. convergence of the series (1.1), can be preserved for (1.6) in addition to the uniform convergence of ${}_vS_{n_k}(f)$ to $f$.

In Section 7 we deduce some results which show that rearranging a Fourier series has some positivity properties. Several theorems could be formulated concerning local behaviour, but we mention only one, just in connection with this positivity character. Our Theorem 7 is to be compared to Gibbs' phenomenon, as formulated, e.g., in [10, Vol. I, p. 61].

Last, we apply our results to the solution of a certain extremal problem, in Section 8. It shows very clearly the connection to Gibbs' phenomenon, but it is not only a mere illustration. It emerged in [8], in connection with a certain problem of analytic number theory concerning prime distribution and the Riemann zeta function, and the solution has interesting consequences.

The main idea of the present work can be applied to more general situations, such as $d$-dimensional periodic functions, uniformly almost periodic functions [9], or Fourier series of general orthonormal systems. We return to these elsewhere.

## 2

In the introduction we formulated our Theorem 1. Since $C(T) \subset L^2(T)$ and $V_{n,n}(f) \to f$ uniformly for $f \in C(T)$, Theorem 1 follows immediately from the following

THEOREM 2. *Let $f \in L^2(T)$. We can find sequences $n_k \leqslant N_k \leqslant 2n_k \to \infty$ and a permutation $v$ for which*

$$v(j) \in [j/2, 2j] \tag{2.1}$$

*and*

$$\|{}_vS_{N_k} - V_{n_k,n_k}\|_\infty \leqslant (\log\log n_k)^{-1/2}. \tag{2.2}$$

*Proof of Theorem* 2. Introduce the notation

$$c_k := \sqrt{a_k^2 + b_k^2}, \tag{2.3}$$

where $a_k$, $b_k$ are the coefficients in (1.1). For proving Theorem 2, we need the following lemma, which, however, will be used later, too.

LEMMA 1. *Let* $f \in L(T)$ *have Fourier series* (1.1) *with the property that for some* $\eta(k) > 0$

$$\log n_k \sum_{j=n_k+1}^{n_k+m_k} c_j^2 \leqslant \eta(k) \qquad (k \in \mathbb{N}) \tag{2.4}$$

*hold with* $n_k, m_k \in \mathbb{N}$, $m_k \leqslant n_k < n_k + m_k \leqslant n_{k+1}$ ($c_j$ *is defined in* (2.3)). *Then there exist a rearrangement* $\nu$ *and a sequence of natural numbers* $N_k \in [n_k, n_k + m_k]$ *for which we have*

$$\begin{aligned} j \in (n_k, n_k + m_k] &\Leftrightarrow \nu(j) \in (n_k, n_k + m_k], \\ j \in (n_k + m_k, n_{k+1}] &\Rightarrow \nu(j) = j, \end{aligned} \tag{2.5}$$

*and*

$$\|{}_\nu S_{N_k}(f) - V_{n_k, m_k}(f)\|_\infty \leqslant 8\sqrt{\eta(k)} \qquad (k \in \mathbb{N}). \tag{2.6}$$

First of all, let us deduce Theorem 2 from this lemma. According to Lemma 1, it suffices to find a sequence $n_k \to \infty$ with

$$\log n_k \sum_{j=n_k+1}^{2n_k} c_j^2 \leqslant \frac{1}{64 \log \log n_k} =: \eta(k).$$

If it does not work, then for $n > n_0$ we must have

$$\sum_{j=n+1}^{2n} c_j^2 > \frac{1}{64 \log n \log \log n};$$

consequently by $f \in L^2(T)$ and the above inequality

$$\begin{aligned} \infty > \sum_{n_0+1}^{\infty} c_j^2 &= \sum_{l=0}^{\infty} \sum_{j=2^l n_0+1}^{2^{l+1} n_0} c_j^2 \\ &> \sum_{l=1}^{\infty} \frac{1}{64(\log n_0 + l)(\log n_0 + \log l)}, \end{aligned}$$

a contradiction in view of the divergence of $\sum (l \log l)^{-1}$. For proving Lemma 1, the essential part is

LEMMA 2. *Let* $\eta > 0$, $m, n \in \mathbb{N}$, *be arbitrary with* $5 < n$, $m \leqslant n$, *and* $f \in L(T)$. *If*

$$\sum_{k=n+1}^{n+m} c_k^2 < \frac{\eta}{\log n}, \tag{2.7}$$

*then there exists some* 0–1 *sequence* $\omega = (\omega_1, ..., \omega_m)$ *in* $\{0, 1\}^m$ *for which*

$$\left\| S_n(f) + \sum_{k=1}^{m} \omega_k A_{n+k} - V_{n,m}(f) \right\|_\infty < 8\sqrt{\eta}. \tag{2.8}$$

We postpone the proof of Lemma 2 to the next section, but prove that Lemma 1 follows from Lemma 2. Indeed, in Lemma 2 take $n = n_k$, $m = m_k$, $\eta = \eta(k)$. Define $M_k := \omega_1 + \cdots + \omega_m$ for a 0–1 sequence $\omega$ provided by Lemma 2. For any $v\colon \mathbb{N} \leftrightarrow \mathbb{N}$ satisfying (2.5) and

$$v(j) \in (n_k, n_k + M_k] \Leftrightarrow \omega_j = 1 \tag{2.9}$$

we get with $N_k := n_k + M_k$ that

$$_vS_{N_k}(f) = S_{n_k}(f) + \sum_{1}^{m_k} \omega_i A_{n_k+i}. \tag{2.10}$$

Now (2.8) is just identical with (2.6) in view of (2.10).

## 3

We begin with two propositions.

PROPOSITION 1 (Bernstein's Inequality). *Let X be any random variable,* $\varepsilon > 0$ *and* $\lambda > 0$ *any parameters. With P denoting probability and E expectation we have then*

$$P(|X - E(X)| \geqslant \varepsilon) \leqslant e^{-\varepsilon\lambda}\{E(e^{\lambda(X - E(X))}) + E(e^{\lambda(E(X) - X)})\}.$$

*Proof.* For any random variable $Y$ trivially

$$P(Y \geqslant \varepsilon) = e^{-\varepsilon\lambda} \int_{\{Y \geqslant \varepsilon\}} e^{\varepsilon\lambda}\, dP$$

$$\leqslant e^{-\varepsilon\lambda} \int_{\{Y \geqslant \varepsilon\}} e^{\lambda Y}\, dP \leqslant e^{-\varepsilon\lambda} E(e^{\lambda Y}).$$

Applying this to $Y = X - E(X)$ and $Y = E(X) - X$ together gives Proposition 1.

PROPOSITION 2. *For any* $z \in \mathbb{C}$ *and* $0 \leqslant a \leqslant 1$ *we have*

$$|ae^{(1-a)z} + (1-a)e^{-az}| \leqslant e^{|z|^2}.$$

*Proof.* For $|z|>1$ this is trivial; otherwise the Taylor series expansion of $e^{(1-a)z}$ and $e^{-az}$ gives with some $|r_j|<1$ $(j=1,2)$

$$\begin{aligned}
&|ae^{(1-a)z}+(1-a)\,e^{-az}| \\
&\quad=\left|a\left\{1+(1-a)z+\frac{(1-a)^2z^2}{2}+r_1\frac{e(1-a)^3z^3}{6}\right\}\right. \\
&\qquad\left.+(1-a)\left\{1-az+\frac{a^2z^2}{2}+r_2\frac{ea^3z^3}{6}\right\}\right| \\
&\quad\leqslant 1+\frac{(1-a)a}{2}|z|^2+\frac{e}{6}|z|^2 \\
&\quad\leqslant 1+|z|^2\leqslant e^{|z|^2}.
\end{aligned}$$

*Proof of Lemma* 2. Let us write

$$\Omega=\{0,1\}^m=\{\omega=(\omega_1,...,\omega_m)\},\qquad \mathscr{A}=2^\Omega \tag{3.1}$$

and for $A\subset\Omega$ with the weights (1.3),

$$\begin{gathered}
P(A):=\sum_{\omega\in A}P(\omega),\qquad P(\omega):=\prod_{k=1}^m P_k(\omega_k), \\
P_k(\omega_k):=\begin{cases}p_k:=p_{n+k}^{(n,m)}, & \omega_k=1\\ 1-p_k=1-p_{n+k}^{(n,m)}, & \omega_k=0.\end{cases}
\end{gathered}\tag{3.2}$$

It is plain that $(\Omega,\mathscr{A},P)$ is a probability space, and it suffices to show that the probability of the "event" (2.8) is positive. At first we establish some properties of this probability space. By the definition (3.2) the coordinate projections

$$X_k\colon\Omega\to\{0,1\},\qquad X_k(\omega):=\omega_k$$

are totally independent random variables, and so for any fixed $x\in T$ even the random variables

$$\begin{gathered}
Y_k(x,\ )\colon\Omega\to\mathbb{R}, \\
Y_k(x,\omega):=\omega_kA_{n+k}(x)=X_k(\omega)\,A_{n+k}(x)
\end{gathered}\tag{3.3}$$

are independent. The sum of these independent random variables is

$$F(x,\ ):=\sum_{k=1}^m Y_k(x,\ )=\sum_{k=1}^m X_kA_{n+k}(x),\tag{3.4}$$

and the expectation of this sum is

$$E(x) := E(F(x,\ )) = \sum_{k=1}^{m} p_k A_{n+k}(x)$$
$$= V_{n,m}(f)(x) - S_n(f)(x). \tag{3.5}$$

This means that all that we have to do is estimate the deviation of $F(x,\ )$ from its expectation (3.5) and prove that this difference is uniformly small with a positive probability.

For any fixed $x \in T$ and for any parameters $\varepsilon > 0$, $\lambda > 0$ we get by Bernstein's inequality (Proposition 1) that

$$P(|F(x,\ ) - E(x)| \geqslant \varepsilon)$$
$$\leqslant e^{-\lambda\varepsilon}\{E(e^{\lambda(F(x,\ ) - E(x))}) + E(e^{\lambda(E(x) - F(x,\ ))})\}. \tag{3.6}$$

Since $F(x,\ )$ is a sum of the independent variables (3.3) these expressions can be decomposed, and by means of Proposition 2 with $z = \lambda A_{n+k}(x)$ and $a = p_k$, they can be estimated as well. We get

$$E(e^{\lambda(F(x,\ ) - E(x))}) = \prod_{k=1}^{m} E(e^{\lambda(Y_k(x,\ ) - E(Y_k(x,\ )))})$$
$$= \prod_{k=1}^{m} \{p_k e^{(1-p_k)\lambda A_{n+k}(x)} + (1-p_k)\, e^{-p_k \lambda A_{n+k}(x)}\}$$
$$\leqslant \prod_{k=1}^{m} e^{\lambda^2 A^2_{n+k}(x)} \leqslant \prod_{k=1}^{m} e^{\lambda^2 c^2_{n+k}} = e^{\lambda^2 \sum_{k=1}^{m} c^2_{n+k}} < e^{\lambda^2 \eta/\log n}, \tag{3.7}$$

and similarly

$$E(e^{\lambda(E(x) - F(x,\ ))}) \leqslant e^{\lambda^2 \eta/\log n}, \tag{3.8}$$

so from (3.6), (3.7), and (3.8)

$$P(|F(x,\ ) - E(x)| \geqslant \varepsilon) \leqslant 2\exp(\lambda^2 \eta/\log n - \varepsilon\lambda). \tag{3.9}$$

For any finite point set

$$\{x_l : l = 1, ..., L\}\ (\subset T) \tag{3.10}$$

it follows from (3.9) trivially that

$$P(|F(x_l,\ ) - E(x_l)| < \varepsilon (l = 1, ..., L))$$
$$\geqslant 1 - 2L\exp(\lambda^2 \eta/\log n - \varepsilon\lambda). \tag{3.11}$$

Fixing the parameters as

$$L = 13n, \qquad \lambda = (\log n)/\sqrt{\eta}, \qquad \varepsilon = 4\sqrt{\eta}, \tag{3.12}$$

and calling an $\omega \in \Omega$ "good" if it satisfies the left of (3.11), we infer for $n > 5$ that

$$\begin{aligned} P(\omega \text{ is good}) &\geqslant 1 - 26n \exp(\log n - 4 \log n) \\ &= 1 - 26n^{-2} > 0. \end{aligned} \tag{3.13}$$

The only thing to do now is choose the nodes in (3.10) so as to guarantee that for any good $\omega \in \Omega$ even (2.8) is satisfied. This can be done by taking equidistant nodes $x_l = 2\pi l/L$ ($l = 1, ..., L$). Really, by Bernstein's well-known inequality concerning the maximum-norm of a trigonometric polynomial and its derivative, we get for any $\omega \in \Omega$ and $x \in T$

$$\begin{aligned} |F(x, \omega) - E(x)| &= |F(x_{l_0}, \omega) - E(x_{l_0}) \\ &\quad + \int_{x_{l_0}}^{x} (F'(y, \omega) - E'(y))\, dy| \\ &\leqslant \max_{1 \leqslant l \leqslant L} |F(x_l, \omega) - E(x_l)| \\ &\quad + \frac{\pi}{13n} (n + m) \|F( \,, \omega) - E\|_\infty, \end{aligned} \tag{3.14}$$

where $x_{l_0}$ is the node closest to $x$. From $m \leqslant n$ and (3.14) we get

$$\|F( \,, \omega) - E\|_\infty < 2 \max_{l = 1, ..., L} |F(x_l, \omega) - E(x_l)|. \tag{3.15}$$

Recalling the meaning of $\omega$ being good, (3.15) states by (3.12) that for all good $\omega \in \Omega$

$$\|F( \,, \omega) - E\|_\infty < 2\varepsilon = 8\sqrt{\eta} \tag{3.16}$$

holds. This and (3.13) prove Lemma 2.

*Remark* 1. By varying some constants, we can deduce easily in (3.13) any inequalities with right-hand side $1 - n^{-a}$ for any $a > 0$ and $n > n_0(a)$.

*Remark* 2. On the other hand, for any single $\omega \in \Omega$, when, say, $m = 2\mu$ is even and large, we have

$$P(\omega) \leqslant \left( \frac{(2\mu)(2\mu - 1) \cdot ... \cdot (\mu + 1)}{m^\mu} \right)^2 \frac{1}{2} \sim \left( \frac{2}{e} \right)^m,$$

and so there are at least

$$\frac{1}{2}|\Omega|^{\alpha}, \qquad \alpha=\log_2(e/2)=\frac{1}{\log 2}-1=0.43...$$

good $\omega\in\Omega$.

*Remark* 3. It can be checked that for any finite set $\mathscr{F}=\{f_1, ..., f_K\}$ of functions with corresponding Fourier coefficients $c_{k,j}$ $(k=1, ..., K,$ $j=0, 1, 2, ...)$ as defined in case of $f$ according to (1.1) and (2.3), we have a common $\omega\in\Omega$ in (2.8) whenever (2.7) is true for all the $K$ sequences, i.e.,

$$\sum_{j=n+1}^{n+m} c_{k,j}^2<\frac{\eta}{\log n} \qquad (k=1, ..., K). \tag{3.17}$$

This is obvious from the fact that (for $n>n_0(K)=\sqrt{26K}$ instead of $n>5$ in the condition of Lemma 2) a repetition of the above proof for $k=1, ..., K$ separately gives in place of (3.13)

$$P\ (\omega \text{ is good for } f_k)\geqslant 1-26n^{-2}>1-1/K, \tag{3.18}$$

and hence

$$P\ (\omega \text{ is good for every } f_k\ (k=1, ..., K))>0. \tag{3.19}$$

Repeating the argument proving Theorem 1 from Lemma 1, but for $c_{1,j}^2+\cdots+c_{K,j}^2$ in place of $c_j^2$, we can infer the existence of a common $v$ in Theorems 1 and 2. We note that this will be the case in all our theorems below, but we will not check it explicitly. Only in one case, in the proof of Theorem 4, does one need a further—though simple—trick to obtain this variant of the theorem. We note that as Remark 5 there. As a result, all of our theorems are valid for $f$ and $\tilde{f}$ with common $v$, etc., whenever $\tilde{f}$ is in the same class (e.g., in the case of Theorem 1 if both $f$ and $\tilde{f}$ are continuous).

## 4

As we expressed before, we hope for the truth of the following

*Conjecture.* For any continuous $f$ there exist rearrangements of its Fourier series uniformly convergent to $f$.

EXAMPLE. The well-known example of Fejér of a continuous function with divergent Fourier series is

$$F(x)=\sum_{k=1}^{\infty}\frac{1}{k^2}\sum_{\substack{s=-S_k/2\\ s\neq 0}}^{S_k/2}\frac{\cos((S_k+s)x)}{s} \qquad (S_k=2^{k^3}).$$

This function $F$ satisfies the assertion of the conjecture with

$$v(j)=\begin{cases}2|j-S_k|-2+S_k/2, & S_k<j\leqslant 3S_k/2\\ 2|j-S_k|-1+S_k/2, & S_k/2\leqslant j<S_k.\end{cases}$$

In order to attack the above conjecture, one may start with Theorem 1 and then look for appropriate permutations of the blocks $(N_k, N_{k+1}]$ distinctly. A. A. Sahakian kindly called my attention to the fact that the conjecture would follow immediately from the affirmative solution of the following

*Conjecture'*. There exists an absolute constant $C$ such that for all $N\in\mathbb{N}$ and trigonometric polynomial $T(x)=\sum_{i=1}^{N} A_i(x)$ there is some permutation $\sigma$ of $[1, N]$ for which

$$\max_{n\leqslant N}\left\|\sum_{i=1}^{n} A_{\sigma(i)}\right\|_\infty\leqslant C\|T\|_\infty.$$

However, we can prove the converse, i.e., Conjecture' is in fact equivalent to the foregoing conjecture. Indeed, suppose the existence of trigonometric polynomials

$$T_k=\sum_{i=1}^{N_k} A_i^{(k)},\qquad \|T_k\|_\infty=1,\qquad \min_\sigma\max_{n\leqslant N_k}\|{}_\sigma S_n(T_k)\|_\infty>2^k. \quad (4.1)$$

Let us define

$$f(x)=\sum_{k=1}^{\infty}\frac{1}{k^2}T_k(M_kx),\qquad U_k(x)=\sum_{m=1}^{k}\frac{1}{m^2}T_m(M_mx), \quad (4.2)$$

where $M_k$ are defined by, e.g., $M_1=1$, $M_{k+1}=(N_kM_k)!$. Now, if $v$ is any permutation of $\mathbb{N}$, we have for arbitrary $n$, $h$, and $y$

$$\left|\frac{1}{h}\int_y^{y+h}{}_vS_n(f)\right|\leqslant\|{}_vS_n(f)\|_\infty.$$

If $y$ is chosen to be a maximum-point of $|{}_vS_n(U_k)|$, and $h=2\pi/M_{k+1}$, we get

$$\begin{aligned}\|{}_vS_n(f)\|_\infty &\geqslant\left|\frac{M_{k+1}}{2\pi}\int_y^{y+2\pi/M_{k+1}}{}_vS_n(U_k)+O\right|\\ &\geqslant\|{}_vS_n(U_k)\|_\infty-\frac{2\pi}{M_{k+1}}\|{}_vS_n(U_k)'\|_\infty\\ &\geqslant\|{}_vS_n(U_k)\|_\infty\left(1-\frac{2\pi N_kM_k}{M_{k+1}}\right)\end{aligned}$$

($'$ denoting differentiation), whence by (4.2),

$$\left\|{}_{\nu}S_n\left(\frac{1}{k^2}T_k\right)\right\|_{\infty} \leqslant \|{}_{\nu}S_n(U_k)\|_{\infty} + \|{}_{\nu}S_n(U_{k-1})\|_{\infty} \leqslant 4\|{}_{\nu}S_n(f)\|_{\infty}.$$

In view of (4.1) this last inequality contradicts to the conjecture.

Another equivalent statement can be formulated asserting the existence of an absolute constant $C$ such that

$$\inf_{\nu} \sup_{n} \|{}_{\nu}S_n(f)\|_{\infty} \leqslant C\|f\|_{\infty} \qquad \text{for all} \quad f \in C(T). \tag{4.3}$$

## 5

Let $f \in C(T)$ be given, and denote, as usual, $\omega(f, h) := \sup\{|f(x+t) - f(x)| \colon x, t \in T, |t| \leqslant h\}$ as its uniform modulus of continuity and

$$E_n(f) := \min\left\{\|f - P\|_{\infty} \colon P(x) = \sum_{k \leqslant n} (\alpha_k \cos kx + \beta_k \sin kx)\right\},$$

the $n$th approximation constant for any $n \in \mathbb{N}$. These two quantities are related by

$$E_n(f) \leqslant c_1 \omega\left(f, \frac{1}{n}\right) \leqslant c_2 \frac{1}{n} \sum_{k \leqslant n} E_k(f). \tag{5.1}$$

This means, that if $f$ has a large $E_n(f)$, then it is not too smooth, and if $f$ is not smooth enough, i.e., it has large $\omega(f, \cdot)$, then it can be approximated relatively slowly. For example, we may call a function $f$ "weakly continuous", if[1]

$$E_n(f) \geqslant c_3(\log\log n)^{-1/2}. \tag{5.2}$$

THEOREM 3. *Suppose that $f \in C(T)$ is weakly continuous in the sense of (5.2). Then there exists a permutation $\nu$ satisfying*

$$\liminf_{n \to \infty} \frac{\|f - {}_{\nu}S_{2n}(f)\|_{\infty}}{E_n(f)} \leqslant 4. \tag{5.3}$$

[1] Such functions exist in abundance, since for any $e_n \searrow 0$, $e_n \neq 0$ there exists a continuous function with $E_n(f) = e_n$; see [11].

*Proof.* As is well known, $\|f - V_{n,n}(f)\|_\infty \leqslant 4E_n(f)$, and so Theorem 2 and (5.2) give with a certain $v$

$$\begin{aligned}\|{}_vS_{N_k}(f) - f\|_\infty &\leqslant \|{}_vS_{N_k}(f) - V_{n_k,n_k}(f)\|_\infty + 4E_{n_k}(f)\\ &\leqslant E_{n_k}(f)\left\{4 + \frac{(\log\log n_k)^{-1/2}}{E_{n_k}(f)}\right\}\\ &\leqslant E_{n_k}(f)\{4 + 1/c_3\},\end{aligned} \tag{5.4}$$

proving that the right of (5.3) is at least finite. With a modification of Theorem 2 by taking $\varepsilon'(n) = \sigma(\log\log n)^{-1/2}$ in place of $(\log\log n)^{-1/2}$ and $\eta'(k) = \sigma^2/(64 \log\log n_k)$ in place of $\eta(k)$ we obtain similarly $4 + \sigma/c_3$ in (5.4), so letting $\sigma \to 0$ (which means a diagonal argument in the construction of $v$), we are led to

$$\liminf_{k\to\infty} \frac{\|f - {}_vS_{N_k}(f)\|_\infty}{E_{n_k}(f)} \leqslant 4.$$

Since $N_k$ can be supposed to be even, and $N_k \leqslant 2n_k$, hence $E_{n_k}(f) \leqslant E_{N_k/2}(f)$, and we get (5.3).

PROBLEM 3. Can we take ${}_vS_n(f)$ instead of ${}_vS_{2n}(f)$ in (5.3)?

PROBLEM 4. What is the situation, when $f$ is a smooth function, e.g., if for some $0 < \alpha < 1$, $f \in \mathrm{Lip}(\alpha, \infty)$? Does ${}_vS_n(f)$ approximate $f$ better, or does the same extra $\log n$ factor occur after every rearrangement, too?

## 6

Similarly to the asymptotic equivalence $\sim$ of sequences, we may introduce the notion of "logarithmic equivalence" or "rough equivalence" of sequences as

$$t_k \mathrel{\mathcal{N}} s_k \Leftrightarrow \frac{\log t_k}{\log s_k} \to 1 \qquad (k \to \infty). \tag{6.1}$$

Note that for taking $s_k = t_{k+1}$, we obtain the notion of "slowly increasing sequence" in the sense of Karamata. Using this notation, the following theorem can be formulated.

THEOREM 4. *Let $f \in L^2(T)$. Then there exist some permutation $\pi\colon \mathbb{N} \leftrightarrow \mathbb{N}$*

*satisfying* (2.1), *and some slowly increasing sequences* $n_k \leqslant N_k \leqslant 2n_k \to \infty$, *such that uniformly in* $T$

$$ {}_{\pi}S_{N_k}(f) - V_{n_k, n_k}(f) \to O, \tag{6.2}$$

*and*

$$ {}_{\pi}S_j(f)(x) \to f(x) \qquad \textit{for} \quad \textit{a.a. } x \in T. \tag{6.3}$$

*Proof.* Our permutation will be defined as the composition of two others,

$$\pi := \sigma \circ \nu. \tag{6.4}$$

The first, $\nu$, provides (6.2) similarly to the preceding theorems, while the second is to give even (6.3), too.

LEMMA 3. *Let* $d_j \geqslant 0$ *be given with* $\sum d_j < \infty$. *Then there exist* $\eta(k) \to 0$ *and a slowly increasing* $n_k \to \infty$ *with* $n_{k+1} \geqslant 2n_k$ *such that*

$$\log n_k \sum_{n_k+1}^{2n_k} d_j < \eta(k) \qquad (k \in \mathbb{N}). \tag{6.5}$$

*Proof of Lemma* 3.

$$n_{k+1} := \min \left\{ n \geqslant 2n_k : \log n \sum_{n+1}^{2n} d_j < \eta(k+1) := \left( \sum_{2n_k+1}^{\infty} d_j \right)^{1/2} \to 0 \right\}. \tag{6.6}$$

Now for any fixed $A > 1$ in case of $n_{k+1} > n_k^A$ we obtain

$$\begin{aligned} \eta^2(k+1) &= \sum_{2n_k+1}^{\infty} d_j \geqslant \sum_{l=1}^{[\log_2(n_{k+1}/n_k)]-1} \sum_{2^l n_k+1}^{2^{l+1}n_k} d_j \\ &> \sum_{l=1}^{\log(n_{k+1}/n_k)} \frac{\eta(k+1)}{\log(2^l n_k)} \\ &> \eta(k+1) \int_1^{\log(n_{k+1}/n_k)} \frac{dt}{\log(n_k)+t} \\ &= \eta(k+1) \log\left( \frac{\log n_{k+1}}{\log n_k + 1} \right) \\ &> \eta(k+1) \frac{1}{2} \log A. \end{aligned} \tag{6.7}$$

Clearly this is a contradiction for large $k$ by $\eta(k)\to 0$, so for $k>k_0(A)$, $n_{k+1}\leqslant n_k^A$, whence $n_k$ is slowly increasing, and by definition it satisfies (6.5), too.

*Remark* 4. If we do not want to have a slowly increasing $n_k$, then the argument in Section 2 proving Theorem 2 may replace Lemma 3.

Now turning to the construction of $\nu$, we apply with $d_j=c_j^2$ the above Lemma 3. So we get a slowly increasing $n_k\to\infty$ with $2n_k\leqslant n_{k+1}$, and a sequence $\eta(k)\to 0$, for which (6.5) holds. Choosing $m_k=n_k$ in (2.4), (6.5) is just identical with it, and so Lemma 1 gives a $\nu$ and some $N_k$ satisfying $n_k\leqslant N_k\leqslant 2n_k$ with (2.6) (for $m_k=n_k$) and so

$$ {}_\nu S_{N_k}(f)-V_{n_k,n_k}(f)\to O \qquad \text{uniformly in } T \tag{6.8}$$

follows. Moreover, here $n_k$ and $N_k$ satisfy all that we need, and $\nu$ is subject to the conditions (2.5).

We want to define a $\sigma$ possessing the properties

$$ j\in[2n_k, n_{k+1}]\Rightarrow\sigma(j)=j \tag{6.9}$$

and

$$ \begin{aligned} &j\in(n_k, N_k]\Leftrightarrow\sigma(j)\in(n_k, N_k]\\ &j\in(N_k, 2n_k)\Leftrightarrow\sigma(j)\in(N_k, 2n_k). \end{aligned} \tag{6.10}$$

If $\sigma$ is such, then (2.5) with $m_k=n_k$ and (6.9) give

$$ j\in[2n_k, n_{k+1}]\Rightarrow\pi(j)=j, \tag{6.11}$$

and so $\pi$ satisfies (with $\pi$ in place of $\nu$) (2.1), as stated. Further, (6.4), (6.8), and (6.10) lead to (6.2) at once. So the only thing to do is to define a $\sigma$ subject to (6.9) and (6.10) and giving (6.3). We define the set

$$ H:=\{n\in\mathbb{N}\colon \exists k\in\mathbb{N},\ n=N_k \text{ or } n\in[2n_k, n_{k+1}]\}. \tag{6.12}$$

Since $f\in L^2(T)$, Carleson's theorem states $S_n(f)(x)\to f(x)$ for a.a. $x\in T$. Combining this with (6.4), (6.8), (6.10), (6.11), and (6.12) we obtain, numbering $H$ in increasing order as $h_k$, that

$$ {}_\pi S_{h_k}(f)(x)={}_\nu S_{h_k}(f)(x)\to f(x) \qquad \text{for} \quad \text{a.a. } x\in T. \tag{6.13}$$

Define

$$ \phi_j(x):=\frac{1}{\sqrt{2\pi}\,c_{\nu^{-1}(j)}}A_{\nu^{-1}(j)}(x), \tag{6.14}$$

then $\phi = \{\phi_j : j \in \mathbb{N}\}$ is an ONS in $L^2(T)$, and the Fourier series of $f$ with respect to $\phi$ is just

$$f \sim \sum_j \sqrt{2\pi}\, c_{\nu^{-1}(j)} \phi_j =: \sum_j C_j \phi_j. \tag{6.15}$$

Now by (6.13), (6.14), and (6.15) we have

$$S^{\phi}_{h_k}(f)(x) \to f(x) \qquad \text{for} \quad \text{a.a. } x \in T. \tag{6.16}$$

By the definition of $H$ in (6.12), our requirements (6.9) and (6.10) are identical with

$$\sigma := \bigcup_{k \in \mathbb{N}} \sigma^{(k)}, \tag{6.17}$$

and

$$\sigma^{(k)} : (h_k, h_{k+1}] \leftrightarrow (h_k, h_{k+1}]. \tag{6.18}$$

At this point we may forget all the antecedents and concentrate on the problem as if we knew only that $\phi$ is an ONS and $f \in L^2(T, dx)$ with Fourier series (6.15) and satisfying, for a certain $h_k \nearrow \infty$, (6.16). This is just the general situation in which Garsia's theorem concerning a.e. convergent rearrangements of Fourier series was proved. In fact, Garsia proves a general inequality (see (3.6.16) in [3]), asserting in our setting that for a certain absolute constant $C$ we have

$$\frac{1}{(h_{k+1} - h_k)!} \sum_{\sigma^{(k)} : (h_k, h_{k+1}] \leftrightarrow (h_k, h_{k+1}]} \int_T \max_{h_k < m \leqslant h_{k+1}} \left( \sum_{j = h_k + 1}^{m} C_{\sigma^{(k)}(j)} \phi_{\sigma^{(k)}(j)} \right)^2$$
$$\leqslant C \sum_{j = h_k + 1}^{h_{k+1}} C_j^2.$$

We note that in Garsia's notation our $\sigma^{(k)}$ corresponds to $\sigma$, $h_{k+1} - h_k$ and $m$ correspond to $n$ and $\nu$, $L^2(T, dx)$ corresponds to $L^2(\Omega_1\, d\mu)$, $C_j$ corresponds to $a_{j-h_k}$, and $\phi_j$ to $\phi_{j-h_k}$. In view of Garsia's inequality we can select for each $k$ a permutation $\sigma^{(k)}$ satisfying (6.18) so that

$$\int_T \max_{h_k < m \leqslant h_{k+1}} \left( \sum_{j = h_k + 1}^{m} C_{\sigma^{(k)}(j)} \phi_{\sigma^{(k)}(j)} \right)^2 \leqslant C \sum_{j = h_k + 1}^{h_{k+1}} C_j^2.$$

Summing up these inequalities for $k = 1, 2, \ldots$, $f \in L^2$, Beppo Levi's theorem and (6.16) entail that the permutation $\sigma$ in (6.17) will satisfy

$$_{\sigma}S^{\Phi}_j(f)(x) \to f(x) \qquad \text{for} \quad \text{a.a. } x \in T.$$

This and (6.14)–(6.18) ensure our statement (6.3) with $\pi$ defined in (6.4).

*Remark* 5. If we have a finite set $\{f_1, ..., f_K\}$ of functions belonging to $L^2(T)$, we can find a common $\pi$ to them. Indeed, in view of Remark 3 we have to find a common $\sigma$ only. But for the new function

$$F(x) := f_k(x - 2k\pi) \qquad \left(k = \left[\frac{x}{2\pi}\right]\right) \; x \in [0, 2K\pi)$$

and the new ONS

$$\phi_j(x) = \frac{1}{\sqrt{2\pi}\, c_{k,\nu^{-1}(j)} K} A_{k,\nu^{-1}(j)}(x) \qquad \left(k = \left[\frac{x}{2\pi}\right]\right) \; x \in [0, 2K\pi),$$

a similar application of Garsia's theorem suffices. In particular we have in Theorem 4 the analogues of (6.2) and (6.3) for $\tilde{f}$ (the conjugate of $f$), too, since $\tilde{f} \in L^2(T)$.

*Remark* 6. By sacrificing $j/2 \leqslant \pi(j) \leqslant 2j$ and requiring only $\pi(j) \sim j$, we may apply a further permutation $\tau$ after $\sigma \circ \nu$ with

$$\tau = \bigcup_{k \in \mathbb{N}} \tau^{(k)}, \qquad \tau^{(k)}: (N_k, N_{k+1}] \leftrightarrow (N_k, N_{k+1}].$$

Using this $\tau$ we may push the terms $A_k(x)$ with little coefficients to the beginning of each block $(N_k, N_{k+1}]$, and, since ${}_{\sigma \circ \nu}S_{N_k}(f)$ is close to $V_{n_k, n_k}(f)$, we can fatten the set of indices for which ${}_\pi S_{i_j}(f)$ is close to some $V_{l_j, l_j}(f)$. In this way it can be proved that uniformly in $T$

$${}_\pi S_{i_j}(f) - V_{l_j, l_j}(f) \to 0$$

and also ${}_\pi S_j(f)(x) \to f(x)$ for a.a. $x \in T$, where

$$\sum_{i_j < x} 1 > \sqrt{x}.$$

THEOREM 5. *For any $f \in C(T)$ there exist a rearrangement $\pi$ satisfying*

$$\pi(j) \sim j$$

*and a slowly increasing sequence $N_k \nearrow \infty$, such that*

$${}_\pi S_{N_k}(f) \to f \qquad \textit{uniformly in } T$$

*and*

$${}_\pi S_j(f)(x) \to f(x) \qquad \textit{for a.a. } x \in T.$$

*Proof.* We choose an $m_k < n_k$ instead of $m_k = n_k$ such that $m_k/n_k \to 0$ but $V_{n_k, m_k}(f) \to f$ uniformly in $T$. This can be done, since for any $n, m \in \mathbb{N}$

$$\|f - V_{n,m}(f)\|_\infty \leqslant 2 \frac{n+m+1}{m+1} E_n(f), \tag{6.19}$$

as is well known (see, e.g., [12, pp. 34–35]), and from this point on the proof is identical with that of Theorem 4. Since here $n_k/(n_k+m_k) \leqslant \pi(j)/j \leqslant (n_k+m_k)/n_k$, instead of (2.1) we obtain $\pi(j)/j \to 1$.

## 7

LEMMA 4. *Let $f \in L^2(T)$. Then there exist a rearrangement $\pi$ and sequences $n_k$, $m_k$, $N_k \to \infty$ satisfying $n_k \leqslant N_k \leqslant n_k + m_k < n_{k+1}$ and $m_k/n_k \to \infty$, for which* (6.3) *and*

$$_\pi S_{N_k}(f) - V_{n_k,m_k}(f) \to 0 \qquad (\textit{uniformly in } T) \tag{7.1}$$

*hold.*

*Proof.* We can prove this lemma similarly to Theorem 4, if we start instead of Lemma 3 with

LEMMA 5. *Let $d_j \geqslant 0$ be given with $\sum d_j < \infty$. Then there exist $m_k, n_k \to \infty$ with $m_k/n_k \to \infty$, $n_{k+1} \geqslant n_k + m_k$ and $\eta(k) \to 0$ such that $n_k$ is slowly increasing, and*

$$\log(n_k + m_k) \sum_{n_k+1}^{n_k+m_k} d_j < \eta(k) \qquad (k \in \mathbb{N}). \tag{7.2}$$

*Proof.* One may define $L_k := (n_k + m_k)/n_k$ and use calculations similar to those in the proof of Lemma 3 (with $L_k$ in place of 2). For sufficiently slow $L_k \to \infty$ we arrive at the contradiction

$$\eta^2(k+1) > \frac{\eta(k+1)}{\log L_k} \log\left(\frac{\log n_{k+1}}{\log n_k + \log L_k}\right)$$
$$> \frac{\eta(k+1)}{\log L_k} \log\left(A - \frac{A \log L_k}{\log n_k}\right),$$

supposing that $n_{k+1} > n_k^A$.

THEOREM 6. *Let $f \in L^\infty(T)$. Then there exist a rearrangement $\nu$ and a sequence $N_k \to \infty$ for which*

$$_\nu S_j(f)(x) \to f(x) \qquad \textit{for} \quad \textit{a.a. } x \in T \tag{7.3}$$

*and*

$$\| _\nu S_{N_k}(f) \|_\infty \leqslant (1 + o(1)) \|f\|_\infty. \tag{7.4}$$

*Proof.* By the positivity of the Fejér kernel, we have by (1.4)

$$\|V_{n_k,m_k}(f)\|_\infty \leqslant \frac{n_k+m_k}{m_k}\|\sigma_{n_k+m_k}(f)\|_\infty + \frac{n_k}{m_k}\|\sigma_{n_k}(f)\|_\infty$$
$$\leqslant \left(1+\frac{2n_k}{m_k}\right)\|f\|_\infty. \tag{7.5}$$

Applying Lemma 4 and so $n_k/m_k \to 0$ we obtain the theorem.

THEOREM 7. *Suppose that $f \in L^2(T)$ and for some $x_0 \in T$, $f$ has a jump (i.e., $f(x_0+0)$ and $f(x_0-0)$ exist but differ). Then there exist some permutation $\pi$ and a sequence $N_k \to \infty$, such that for any $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon) > 0$ and $k_0 = k_0(\varepsilon) > 0$ with the property that for $k > k_0$*

$$\max_{|x-x_0|\leqslant\delta} |{}_\pi S_{N_k}(f)(x)|$$
$$\leqslant (1+\varepsilon)\max\{|f(x_0+0)|, |f(x_0-0)|\}. \tag{7.6}$$

*Proof.* We use a localization argument. By the well-known properties of the Fejér kernel, (7.5) can be changed to

$$\max_{|x-x_0|\leqslant\delta} |V_{n,m}(f)(x)|$$
$$\leqslant \left(1+2\frac{n}{m}\right)\left(\max_{|x-x_0|\leqslant\delta}|f(x)| + \frac{c}{\delta^2 n}\int_T |f|\right). \tag{7.7}$$

Using Lemma 4 and (7.7) we can deduce (7.6) easily.

## 8

In the work [8], from a certain problem of analytic number theory we were led to the following extremal problem. Determine $C$, $C'$, and $C^*$, where

$$C(n) := \inf\left\{\|T\|_\infty : T(x) = \sum_{k=1}^n \frac{a_k \sin \lambda_k x}{\lambda_k};\right.$$
$$\left. a_1 = 1,\ \lambda_1 = 1,\ a_k \in \mathbb{N},\ \lambda_k \in \mathbb{R}\right\},$$
$$C'(n) := \inf\left\{\|T\|_\infty : T(x) = \sum_{k=1}^n \frac{a_k \sin kx}{k};\ a_1 = 1,\ a_k \in \mathbb{N}\right\}, \tag{8.1}$$
$$C^*(n) := \inf\left\{\|T\|_\infty : T(x) = \sum_{k=1}^n \frac{a_k \sin(2k-1)x}{2k-1};\ a_1 = 1,\ a_k \in \mathbb{N}\right\},$$

and

$$C = \lim_{n \to \infty} C(n), \qquad C' = \lim_{n \to \infty} C'(n), \qquad C^* = \lim_{n \to \infty} C^*(n). \tag{8.2}$$

Trivially, the sequences (8.1) are nonincreasing, and so the limits in (8.2) exist. Further, since $C^*(n) \geqslant C'(2n) \geqslant C(2n)$, we have $C^* \geqslant C' \geqslant C$. Obviously, for any $T$ in the definition of $C(n)$

$$\|T\|_\infty \geqslant \frac{\int_0^L T(x) \sin x \, dx}{\int_0^L |\sin x| \, dx} \to \frac{\pi}{4} \qquad (L \to \infty). \tag{8.3}$$

On the other hand, for

$$f(x) := \sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{2k-1} \tag{8.4}$$

we have

$$f(x) = (\pi/4) \operatorname{sgn} x \qquad (\operatorname{sgn} x := (-1)^{[x/\pi]}), \ \|f\|_\infty = \pi/4, \tag{8.5}$$

so for infinite sums $C^*(\infty) = \pi/4$. However, for the partial sums

$$S_n(f)(x) = \sum_{k=1}^{n} \frac{\sin(2k-1)x}{2k-1} \tag{8.6}$$

we have only

$$\|S_n(f)\|_\infty \to \frac{1}{2} \int_0^\pi \frac{\sin t}{t} \, dt = 0.92... > \frac{\pi}{4} = 0.78...,$$

which is a characteristic example of Gibbs' phenomenon. So summing up these obvious considerations would only give

$$0.93... \geqslant C^* \geqslant C' \geqslant C \geqslant \pi/4. \tag{8.7}$$

The reason for the gap is just Gibbs' phenomenon, and that is what makes the problem nontrivial. However, by the present method we could solve the problem in [8], proving

THEOREM 8. $C^* = C' = C = \pi/4$.

*Proof.* Knowing (8.7), it suffices to show $C^* \leqslant \pi/4$. Applying Theorem 6 to the function (8.5) with Fourier series (8.4), we are ready.

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*Note added in proof.* In a letter to me, Professor S. V. Konjagin called my attention to the work [15] of P. Uljanov, where pp. 58–59 seem to be the first written source of the rearrangement problem investigated above. Essentially the same example as given in Section 4 was first described by S. Stečkin [14].

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